Lecture 3: Markov Decision Processes

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Introduction to MDPs

- *Markov decision processes* formally describe an environment for reinforcement learning
- Conventionally where the environment is *fully observable*
- i.e. The current *state* completely characterizes the process
- Almost all RL problems can be formalized as MDPs, e.g.
  - Optimal control primarily deals with continuous MDPs
  - Partially observable problems can be converted into MDPs
  - Bandits are MDPs with one state
- References: Sutton & Barto Chapter 3.
  Slides: http://josephmodayil.com

Statistical notation in Reinforcement Learning

- A variety of statistical notation has been used in earlier RL textbooks and earlier versions of this course.
- We will move towards more standard notation.
- Lowercase letters are used for states s and functions f.
- Uppercase letters are used for random variables, S_t, and these are often indexed by time.
- *This is not the case yet for the figures*, so for now assume $V^\pi = v^\pi$ (and similar), particularly in figures. There will be some typos in the slides. Also be aware of this when combining information from different sources.
- An expectation $\mathbb{E}[G_t]$ is often made over future trajectories.
- A probability distribution is denoted by $\mathbb{P}[]$.

Markov Property

"The future is independent of the past given the present"

Consider a sequence of random states, $\{S_t\}_{t \in \mathbb{N}}$, indexed by time.

Definition

A random state S_t has the *Markov* property if and only if $\forall s, s' \in \mathcal{S}$

$$\mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right] = \mathbb{P}\left[S_{t+1} = s' \mid S_1, \ldots, S_t = s\right]$$

for all histories (all instantiations of S_k for $k < t$).

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

State Transition Matrix

For random states with the Markov property, the *state transition probability* is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} = \begin{array}{cc} & \textit{to} \\ \textit{from} & \begin{bmatrix} \mathcal{P}_{11} & \ldots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \ldots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

where each row of the matrix sums to 1.

Markov Process

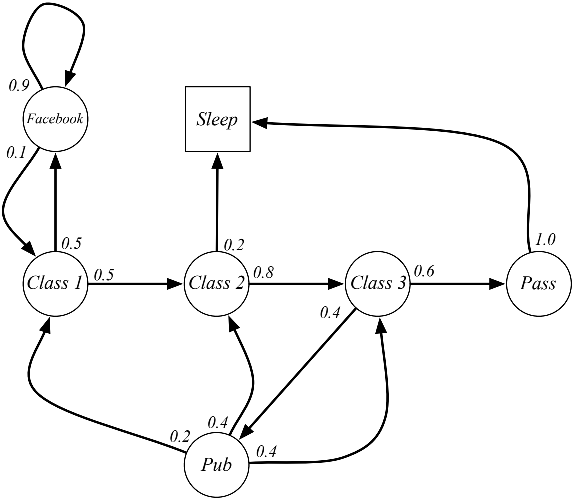
A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, ...$ with the Markov property.

Definition

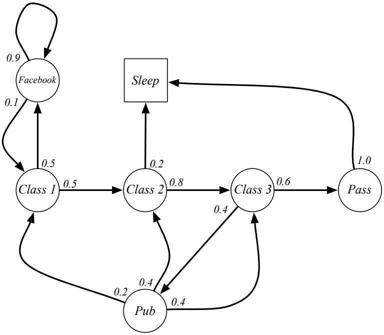
A *Markov Process* (or *Markov Chain*) is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$

Example: Student Markov Chain



Example: Student Markov Chain Episodes

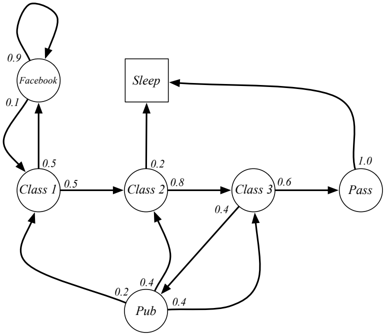


Sample episodes for Student Markov Chain starting from $S_1 = C1$

$$S_1, S_2, ..., S_T$$

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student Markov Chain Transition Matrix



$$\mathcal{P} = \begin{array}{cc} & \begin{array}{ccccccc} C1 & C2 & C3 & Pass & Pub & FB & Sleep \end{array} \\ \begin{array}{c} C1 \\ C2 \\ C3 \\ Pass \\ Pub \\ FB \\ Sleep \end{array} & \begin{bmatrix} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{bmatrix} \end{array}$$

Markov Reward Process

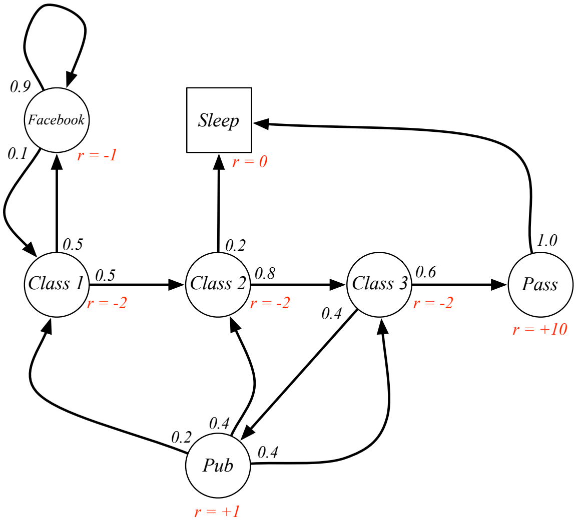
A Markov reward process is a Markov chain with values.

Definition

A *Markov Reward Process* is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$
- $\mathcal{R}$ is a (expected) reward function, $\mathcal{R}_s = \mathbb{E}\left[R_{t+1} \mid S_t = s\right]$
- γ is a discount factor, $\gamma \in [0, 1]$

Example: Student MRP



Return

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behaviour shows preference for immediate reward
- It sometimes possible to use *undiscounted* Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate.

Value Function

The value function $v(s)$ gives the long-term value of state s

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}\left[G_t \mid S_t = s\right]$$

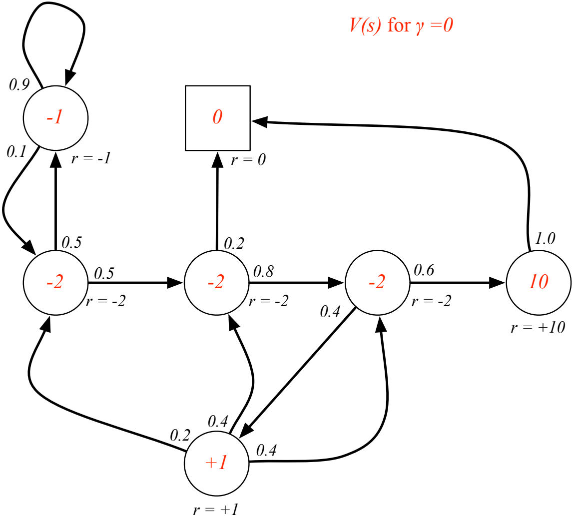
Example: Student Markov Chain Returns

Sample returns for Student Markov Chain:
Starting from $S_1 = \text{C1}$ with $\gamma = \frac{1}{2}$

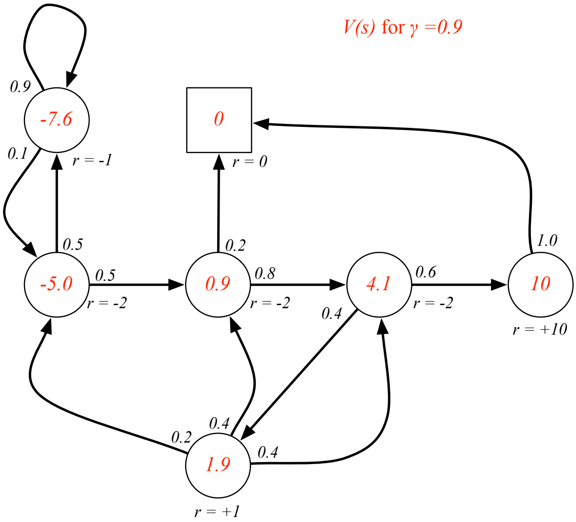
$$G_1 = R_1 + \gamma R_2 + ... + \gamma^{T-1} R_T$$

C1 C2 C3 Pass Sleep	$G_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$G_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$G_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$G_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$=$	-3.20
FB FB FB C1 C2 C3 Pub C2 Sleep			

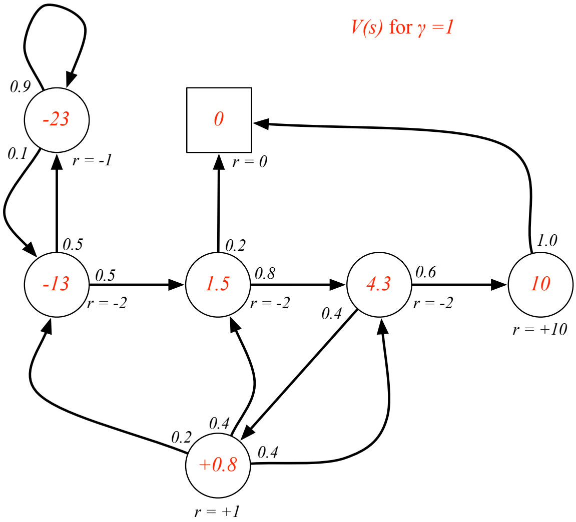
Example: State-Value Function for Student MRP (1)



Example: State-Value Function for Student MRP (2)



Example: State-Value Function for Student MRP (3)



Bellman Equation for MRPs

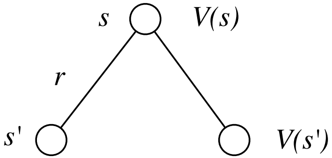
The value function can be decomposed into two parts:

- immediate reward r
- discounted value of successor state $\gamma v(s')$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma\left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

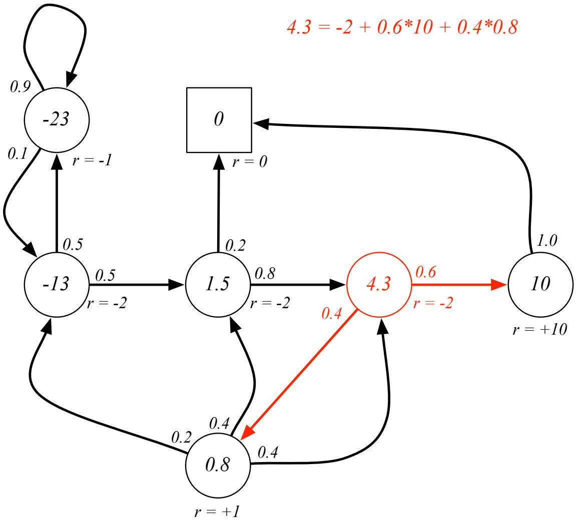
Bellman Equation for MRPs (2)

$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$
\begin{aligned}
v &= \mathcal{R} + \gamma \mathcal{P} v \\
(I - \gamma \mathcal{P})\, v &= \mathcal{R} \\
v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Markov Decision Process

A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all random states are Markov.

Definition

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}^a_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}^a_s = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Markov Decision Process: New notation

A more precise way to describe the state and reward dynamics of a MDP is to give the probability for each possible next state and reward.

$$p(s', r \mid s, a) \equiv \mathbb{P}\left[S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a\right]$$

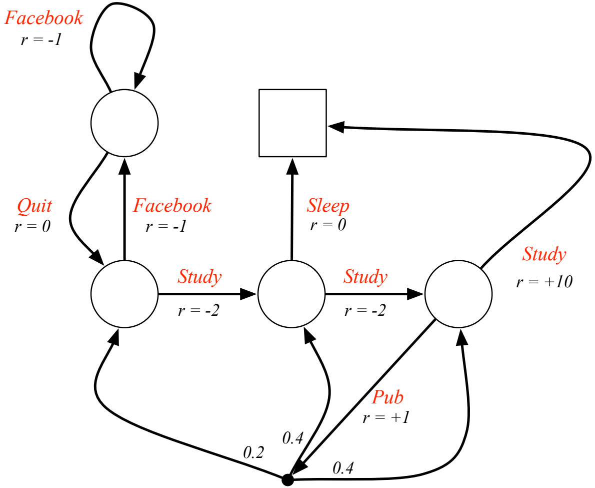
We can then write the state transistion probabilities as

$$\mathcal{P}_{ss'}^{a} \equiv \mathbb{P}\left[S_{t+1} = s' | S_t = s, A_t = a\right] = \sum_{r \in \mathcal{R}} p(s', r | s, a).$$

We can then write the state transistion probabilities as

$$\mathcal{R}_{s}^{a} \equiv \mathbb{E}\left[R_{t+1} = r | S_t = s, A_t = a\right] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r | s, a).$$

Example: Student MDP



Policies (1)

Definition

A *policy* π is a distribution over actions given states,

$$\pi(s, a) = \mathbb{P}[A_t = a \mid S_t = s]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are *stationary* (time-independent), $A_t \sim \pi(S_t, \cdot), \forall t > 0$
- The policy $\pi(s, a)$ is sometimes written as $\pi(a|s)$

Policies (2)

- Given an MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and a policy π
- The state sequence $S_1, S_2, ...$ is a Markov process $\langle \mathcal{S}, \mathcal{P}^{\pi} \rangle$
- The state and reward sequence $S_1, R_2, S_2, ...$ is a Markov reward process $\langle \mathcal{S}, \mathcal{P}^{\pi}, \mathcal{R}^{\pi}, \gamma \rangle$
- where

$$\mathcal{P}^{\pi}_{s,s'} = \sum_{a \in \mathcal{A}} \pi(s,a) \mathcal{P}^{a}_{s,s'}$$
$$\mathcal{R}^{\pi}_{s} = \sum_{a \in \mathcal{A}} \pi(s,a) \mathcal{R}^{a}_{s}$$

Policy Conditional Expectations

Definition

The expectation of a random variable F_t conditional on π being used to select future actions is written as $\mathbb{E}_{\pi}\left[F_t\right]$.

Value Function

Definition

The *state-value function* $v^{\pi}(s)$ of an MDP is the expected return starting from state s, and then following policy π

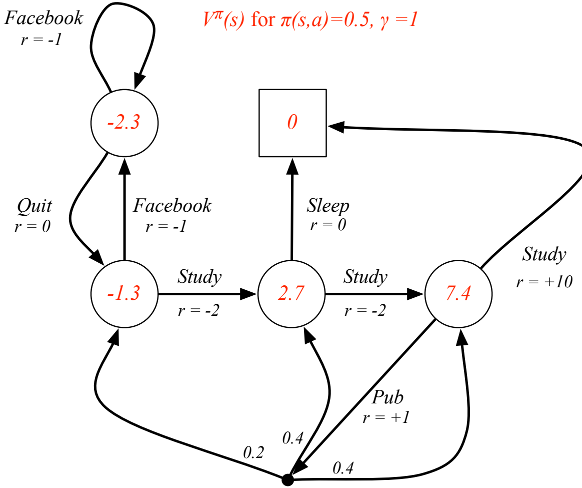
$$v^{\pi}(s) = \mathbb{E}_{\pi}\left[G_t \mid S_t = s\right]$$

Definition

The *action-value function* $q^{\pi}(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q^{\pi}(s, a) = \mathbb{E}_{\pi}\left[G_t \mid S_t = s, A_t = a\right]$$

Example: State-Value Function for Student MDP



Bellman Expectation Equation

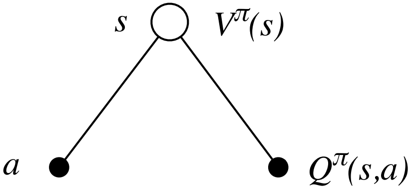
The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v^{\pi}(s) = \mathbb{E}_{\pi}\left[R_{t+1} + \gamma v^{\pi}(S_{t+1}) \mid S_t = s\right]$$

The action-value function can similarly be decomposed,

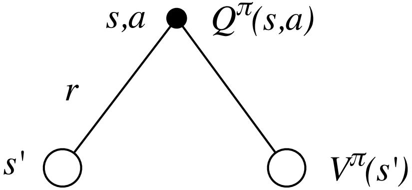
$$q^{\pi}(s,a) = \mathbb{E}_{\pi}\left[R_{t+1} + \gamma q^{\pi}(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

Bellman Expectation Equation for v^π



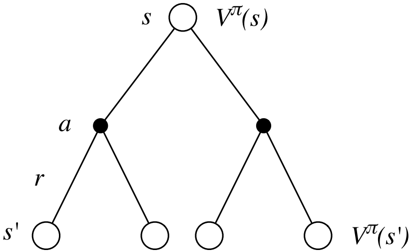
$$v^\pi(s) = \sum_{a \in \mathcal{A}} \pi(s, a) q^\pi(s, a)$$

Bellman Expectation Equation for q^π



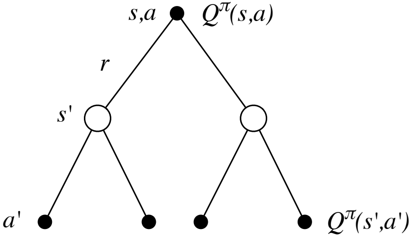
$$q^\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v^\pi(s')$$

Bellman Expectation Equation for v^π (2)



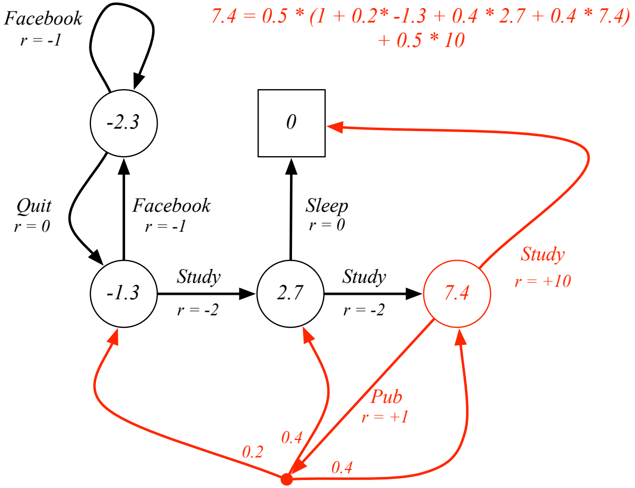
$$v^\pi(s) = \sum_{a \in \mathcal{A}} \pi(s,a) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v^\pi(s') \right)$$

Bellman Expectation Equation for q^π (2)



$$q^\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(s',a') q^\pi(s',a')$$

Example: Bellman Expectation Equation in Student MDP



Bellman Expectation Equation (Matrix Form)

The Bellman expectation equation can be expressed concisely using the induced MRP,

$$v^{\pi} = \mathcal{R}^{\pi} + \gamma \mathcal{P}^{\pi} v^{\pi}$$

with direct solution

$$v^{\pi} = \left(I - \gamma \mathcal{P}^{\pi}\right)^{-1} \mathcal{R}^{\pi}$$

Optimal Value Function

Definition

The *optimal state-value function* $v^*(s)$ is the maximum value function over all policies

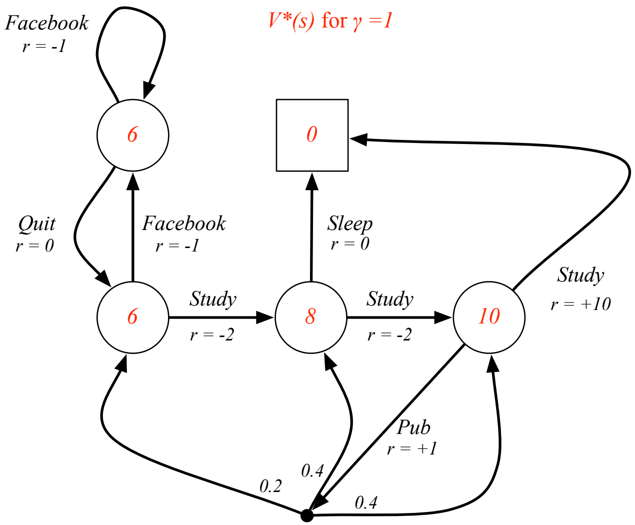
$$v^*(s) = \max_{\pi} v^{\pi}(s)$$

The *optimal action-value function* $q^*(s, a)$ is the maximum action-value function over all policies

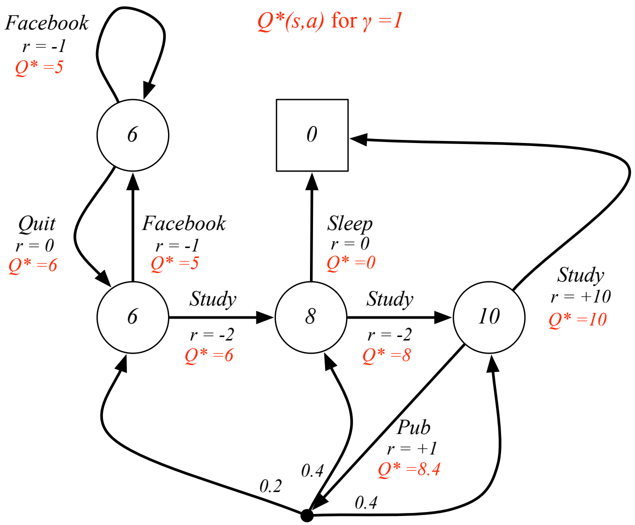
$$q^*(s, a) = \max_{\pi} q^{\pi}(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value fn.

Example: Optimal Value Function for Student MDP



Example: Optimal Action-Value Function for Student MDP



Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v^{\pi}(s) \geq v^{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy π^* that is better than or equal to all other policies, $\pi^* \geq \pi, \forall \pi$*
- *All optimal policies achieve the optimal value function, $v^{\pi^*}(s) = v^*(s)$*
- *All optimal policies achieve the optimal action-value function, $q^{\pi^*}(s, a) = q^*(s, a)$*

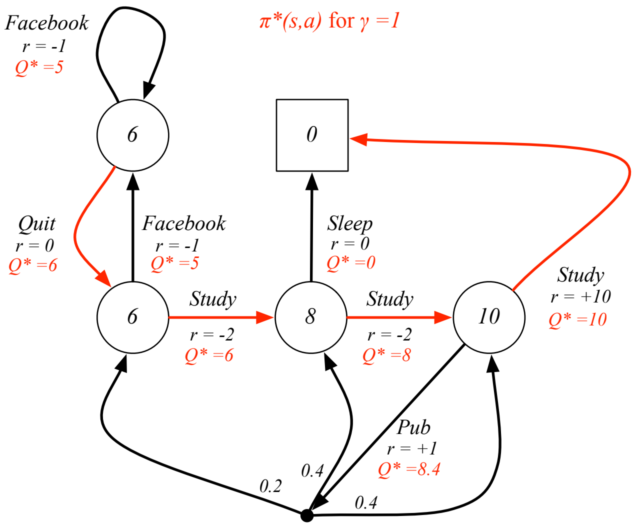
Finding an Optimal Policy

An optimal policy can be found by maximising over $q^*(s,a)$,

$$\pi^*(s,a) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q^*(s,a) \\ 0 & otherwise \end{cases}$$

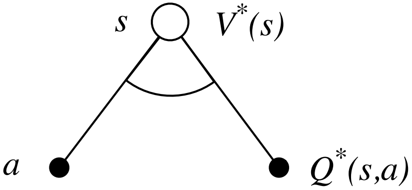
- There is always a deterministic optimal policy for any MDP
- If we know $q^*(s,a)$, we immediately have the optimal policy
- There can be multiple optimal policies

Example: Optimal Policy for Student MDP



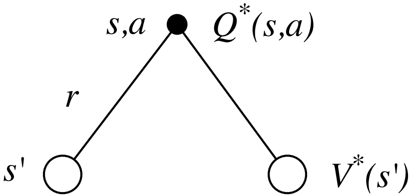
Bellman Optimality Equation for v^*

The optimal value functions are recursively related by the Bellman optimality equations:



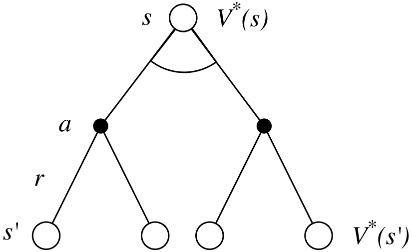
$$v^*(s) = \max_a q^*(s, a)$$

Bellman Optimality Equation for q^*



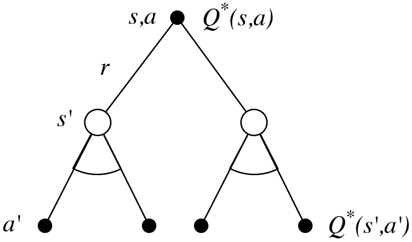
$$q^*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v^*(s')$$

Bellman Optimality Equation for v^* (2)



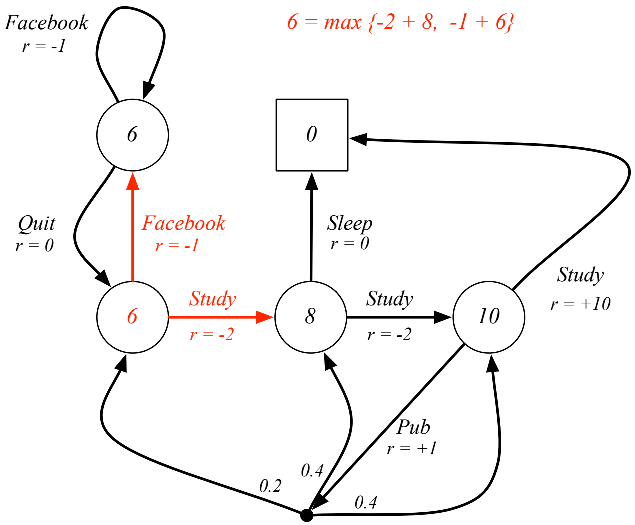
$$v^*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v^*(s')$$

Bellman Optimality Equation for q^* (2)



$$q^*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \max_{a'} v^*(s',a')$$

Example: Bellman Optimality Equation in Student MDP



Solving the Bellman Optimality Equation

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa

Extensions to MDPs

- Infinite and continuous MDPs
- Partially observable MDPs
- Undiscounted, average reward MDPs

Infinite MDPs

The following extensions are all possible:

- Countably infinite state and/or action spaces
  - Straightforward
- Continuous state and/or action spaces
  - Closed form for linear quadratic model (LQR)
- Continuous time
  - Requires partial differential equations
  - Hamilton-Jacobi-Bellman (HJB) equation
  - Limiting case of Bellman equation as time-step $\rightarrow 0$

POMDPs

A POMDP is an MDP with hidden states. It is a hidden Markov model with actions.

Definition

A *Partially Observable Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, \mathcal{P}, \mathcal{R}, \mathcal{Z}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{O}$ is a finite set of observations
- $\mathcal{P}$ is a state transition probability matrix, $\mathcal{P}^a_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}^a_s = \mathbb{E}\left[R_{t+1} = r \mid S_t = s, A_t = a\right]$
- $\mathcal{Z}$ is an observation function, $\mathcal{Z}^a_{s} = \mathbb{P}\left[O_{t+1} = o \mid S_t = s, A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Belief States

Definition

A *history* H_t is a sequence of actions, rewards, and observations

$$H_t = A_1, R_2, O_2, ..., A_{t-1}, R_t, O_t$$

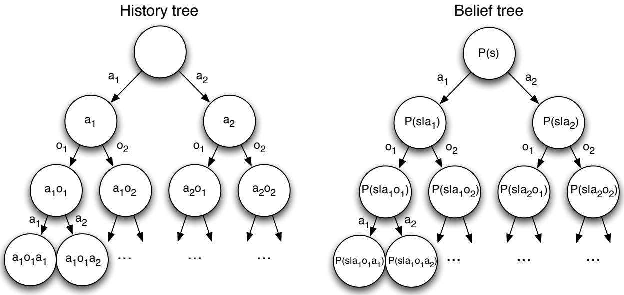
Definition

A *belief state* $b(H_t)$ is a probability distribution over states, conditioned on the history h_t

$$b(H_t) = \left(\mathbb{P}\left[S_t = s^1 \mid H_t\right], ..., \mathbb{P}\left[S_t = s^n \mid H_t\right]\right)$$

Reductions of POMDPs

- The history H_t satisfies the Markov property
- The belief state $b(H_t)$ satisfies the Markov property



- A POMDP can be reduced to an (infinite) history tree
- A POMDP can be reduced to an (infinite) belief state tree

Ergodic Markov Process

An ergodic Markov process is

- *Recurrent*: each state is visited an infinite number of times
- *Aperiodic*: each state is visited without any systematic period

Theorem

An ergodic Markov process has a limiting stationary distribution $d^\pi(s)$ with the property

$$d^\pi(s) = \sum_{s' \in \mathcal{S}} \mathcal{P}^k_{ss'} d^\pi(s')$$

Ergodic MDP

Definition

An MDP is ergodic if the Markov chain induced by any policy is ergodic.

For any policy π, an ergodic MDP has an *average reward per time-step* ρ^{π} that is independent of start state.

$$\rho^{\pi} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{\pi} \left[\sum_{t=1}^{T} R_t \right]$$

Average Reward Value Function

- The value function of an undiscounted, ergodic MDP can be expressed in terms of average reward.
- $\tilde{v}^\pi(s)$ is the extra reward due to starting from state s,

$$\tilde{v}^\pi(s) = \mathbb{E}_\pi \left[\sum_{k=1}^{\infty} (R_{t+k} - \rho^\pi) \mid S_t = s \right]$$

There is a corresponding average reward Bellman equation,

$$\begin{aligned}
\tilde{v}^\pi(s) &= \mathbb{E}_\pi \left[(R_{t+1} - \rho^\pi) + \sum_{k=1}^{\infty} (R_{t+k+1} - \rho^\pi) \mid S_t = s \right] \\
&= \mathbb{E}_\pi \left[(R_{t+1} - \rho^\pi) + \tilde{v}^\pi(S_{t+1}) \mid S_t = s \right]
\end{aligned}$$

Questions?

The only stupid question is the one you were afraid to ask but never did.
-Rich Sutton